

MATRIX MAGIC

by

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Abstract

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With the advent of the proliferation of computers and the institutionalisation of software many aspects of traditional and applied maths are hidden from sight in computer packages churning answers that appear by magic and giving immediate and well-polished results. It is very easy in such circumstances to forget the solid, proven maths machinery that lies behind the achieving of such results. For the mathematician, he may be more interested in the methods of achieving the results than the final application. A full knowledge of how they relate to outcomes contributes to his library of ideas and may be essential for his devising new and advanced solutions.

In this document, I wish by the use of choice examples, initiate the appetite of junior mathematicians and scientists, showing the practical applications of some of their tools, with the full workings exposed for their elucidation. I hope they will be able to use these tools with more confidence in solutions of their own.

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## *Chapter 1*

### APPLICATIONS OF LINEAR ALGEBRA

#### 1.1 Linear equations

Typically, linear equations solve problems of  $n$  variables using  $m \leq n$  linear independent equations, if  $m = n$  then the values of the  $n$  variables are unique.

Example 1.

A buyer at a department store has forgotten the unit costs for women's clothing items provided by a specific supplier, but checking his documentation he finds three invoices detailing items and costs as shown below.

Invoice 1

$$45 \text{ blouses} + 12 \text{ skirts} + 12 \text{ trousers} = \text{£}699.9$$

Invoice 2

$$20 \text{ blouses} + 12 \text{ skirts} + 8 \text{ trousers} = \text{£}421.2$$

Invoice 3

$$20 \text{ blouses} + 10 \text{ skirts} + 5 \text{ trousers} = \text{£}365.5$$

What were the unit costs for Blouses, skirts and trousers?

Example 2.

A surveyor knows he can make a first estimate on the value of a property in direct relation to the number of bedrooms, receptions and bathrooms in the property. He does not know what values to use but he knows the first estimates for three properties as given below.

Property 1

$$1 \text{ bedroom} + 1 \text{ bathroom} = \text{£}120\text{k}$$

Property 2

$$2 \text{ receptions} + 5 \text{ bedrooms} + 2 \text{ bathrooms} = \text{£}735\text{k}$$

Property 3

$$1 \text{ reception} + 3 \text{ bedrooms} + 2 \text{ bathrooms} = \text{£}445\text{k}$$

What first estimate would he make for a block of 5 tenements each having 2 bedrooms and a bathroom that his company wishes to purchase?

The solution of such problems is eased by the creation of an  $n \times n+1$  matrix that we manipulate to reduce into an upper triangular form. The variables are then solved by back substitution.

The two examples given are simple  $n=3$  dimensional problems that are relatively easy to solve manually.

Example 1 can be represented as:

Table 1

line					
1.	45b	+ 12sk	+ 12t	=	699.9
2.	20b	+ 12sk	+ 8t	=	421.2
3.	20b	+ 10sk	+ 5t	=	365.5

Use line 2 to subtract from line 3

Table 2

line					
1.	45b	+ 12sk	+ 12t	=	699.9
2.	20b	+ 12sk	+ 8t	=	421.2
3.-2.	0	-2sk	-3t	=	-55.7

To eliminate blouses from line 2 we subtract  $20/45$  x line 1 from line 2.

Reversing the signs throughout line 2 and line 3 keeps all the resulting elements positive and the result more elegant.

Table 3

line					
1.	45b	+ 12sk	+ 12t	=	699.9
-(2.-(20/45)1.)	0	+ 6.67sk	+ 2.67t	=	110.13
-3.	0	+ 2sk	+ 3t	=	55.7

We eliminate sk in line 3 from by subtracting  $2/6.67$  x line 2 from line 3

Table 4

line					
1.	45b	+ 12sk	+ 12t	=	699.9
2.	0	+ 6.67sk	+ 2.67t	=	110.13
3.-(2/6.67)2.	0	0	+ 2.20t	=	22.68

The solution is given by  $t = 22.68/2.20 = 10.31$

And then back substitution:-

$$6.67 \text{ sk} = 110.13 - 2.67 \times 10.31$$

$$\text{sk} = 12.38$$

$$45 \text{ b} = 699.9 - 123.72 - 148.56$$

$$\text{b} = 427.62/45 = 9.50$$

What is observed doing this relatively simple example, is the quantity of calculations with the possibility of error and how rounding errors carry right through the calculation to give slightly erroneous results, thus the price of skirts,  $\text{sk} = 12.38$  instead of a true value of 12.40 and trousers  $t = 10.31$  instead of a true value of 10.30.

Automating the process would have great value, releasing our need to do tedious calculations and rounding errors can be reduced to a minimum. Using a spreadsheet we can generate an instant result.

Download and open LinEqSoln.xls.

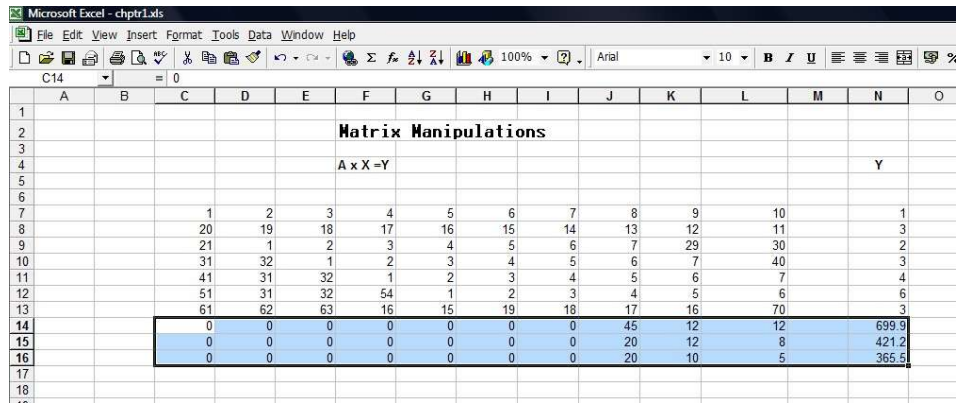
For good programming practice create a copy of this spreadsheet using the ,save as,, option and change its name e.g.  $\text{Example1.xls}$ . On page Linear Equations, at the head of the page we have a 10 x 11 matrix that represents a series of linear equations to be solved. From table 1 we note we may write the formulation of this problem in reduced form:



$$\begin{array}{rclcl}
 45 & 12 & 12 & & = 699.9 \\
 20 & 12 & 8 & & = 421.2 \\
 20 & 10 & 5 & & = 365.5
 \end{array}$$

If we drop the equal signs fill it with leading zeros and make it the last three lines of our opening matrix (make sure you only adjust the lead matrix) as in fig. 1.

Fig 1.



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	
1																
2						<b>Matrix Manipulations</b>										
3																
4						A x X = Y										Y
5																
6																
7			1	2	3	4	5	6	7	8	9	10			1	
8			20	19	18	17	16	15	14	13	12	11			3	
9			21	1	2	3	4	5	6	7	29	30			2	
10			31	32	1	2	3	4	5	6	7	40			3	
11			41	31	32	1	2	3	4	5	6	7			4	
12			51	31	32	54	1	2	3	4	5	6			6	
13			61	62	63	16	15	19	18	17	16	70			3	
14			0	0	0	0	0	0	0	0	45	12	12		699.9	
15			0	0	0	0	0	0	0	0	20	12	8		421.2	
16			0	0	0	0	0	0	0	0	20	10	5		365.5	
17																
18																

We find the 9 following matrices use a systematic Gauss elimination to produce an upper triangular matrix. The 9<sup>th</sup> matrix down holds in the bottom right hand corner a 3 x 4 upper triangular solution of our linear equations. Shown in fig 2.

Fig 2.

	1	2	3	4	5	6	7	8	9	10	
	0	-21	-42	-63	-84	-105	-126	-147	-168	-189	-17
	0	0	21	42	63	84	105	126	168	189	14.19048
	0	0	0	32	64	96	128	160	255	319	19.66667
	0	0	0	0	32	64	96	128	253	317	18.84354
	0	0	0	0	0	54	108	162	422.625	530.625	32.43516
	0	0	0	0	0	0	-5	-10	-12.31944	34.68055511	-6.56136
	0	0	0	0	0	0	0	45	12	12	699.9
	0	0	0	0	0	0	0	0	6.666667	2.666666667	110.1333
									0	-2.2	-22.66

The 10<sup>th</sup> matrix down shows the back substituted solutions our problem, beginning with the last variable and progressing, Shown in fig 3. Thus trousers have a unit cost £10.30

skirts have a unit cost £12.40

blouses have a unit cost £9.50

Fig 3.

		Dimension									
X	DET	1x1	2x2	3x3	4x4	5x5	6x6	7x7	8x8	9x9	10x10
		1	-21	-441	-14112	-451584	-24385536	2.44E+08	10973491387	7.32E+10	-1.6E+11
10		0	0	0	0	0	0	0	0	0	10.3
9		0	0	0	0	0	0	0	0	16.52	12.4
8		0	0	0	0	0	0	0	15.55333333	-11.148	9.5
7		0	0	0	0	0	0	-6.936111	-38.04277779	-69.9356	-70.0047
6		0	0	0	0	0.600651	14.47287	30.02620671	-22.2642	-86.1489	
5		0	0	0	0	0.588861	-0.612442	-7.548553	-7.548552856	79.72064	144.829
4		0	0	0.614583	-0.563138	0.037513	0.037513	0.037513386	0.324319	0.431611	
3		0	0.675737	-0.55343	0.035431	0.035431	0.035431	202.2287642	145.4757	134.5448	
2		0	0.809524	-0.54195	0.072633	0.072633	0.072633	-404.3140334	-289.259	-256.518	
1		1	-0.61905	0.056689	0.056689	0.056689	0.056689	202.2500227	128.4607	2150.557	

Examine sample 2 and we can rationalise the problem into table 5:

Table 5.

bedroom	Bathroom	reception	Value £ 000
1	0	1	= 120
5	2	2	= 735
3	1	2	= 445

We note the second line has a zero for the second element. This could cause problems in the elimination process if it was in the first line our ten line matrix. In such case we would need to manipulate the zero to a lower part of our matrix, preferably below the leading diagonal. We can manipulate the matrix without changing its intrinsic value by changing the position of a complete row or a complete column. If we change the position of a column we have to remember we are changing the order of variables. The reduced solution looks like:

1	0	1	120
5	2	2	735
3	1	2	445

Put this into the bottom right hand side of the opening matrix and filling the left hand cells with zeros.

The results can immediately be seen in matrix 10

Bathroom = 35

Reception = 120

Bedroom = 85

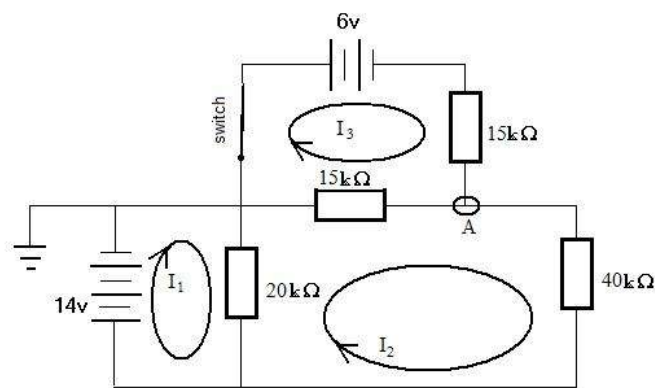
It follows the preliminary estimate for a block of 5 tenements each with two bedrooms and a bathroom based on these values would be

$$10 \times 85 + 5 \times 35 = \text{£}1025\text{k.}$$

### Example 3

Having shown some commercial uses of linear equations, we may now go on to see its use in electric circuit analysis. Consider the circuit shown below in fig 4. What would be the potential at A with the switch closed or alternatively with the switch open?

Fig4.



The secret of answering this problem is in the choice of a minimum number of loop currents that pass through all the major elements of our circuit. We then use Kirchhoff's second law for each loop, i.e. the sum of the voltage drops across each element of the loop equates to the potential difference driving the loop, in other words the sum of the potentials in going round any loop of the circuit from a starting point back to itself is zero.

The current circuit has been analysed into 3 loops, I<sub>1</sub>, I<sub>2</sub> and I<sub>3</sub> as seen.

Using Ohms Law,  $V=IR$  on the three loops we show,

Loop1:

$$20(I_1 - I_2) = -14$$

Loop 2:

$$20(I_2 - I_1) + 15(I_2 - I_3) + 40I_2 = 0$$

Loop3

$$15I_3 + 15(I_3 - I_2) = +6$$

these equations can be rationalised into linear equations of  $I_1$ ,  $I_2$  and  $I_3$ .

$I_1$	$I_2$	$I_3$	V
20	-20	0	= -14
-20	+75	-15	= 0
0	-15	+30	= +6

Entering this matrix in the appropriate fashion into our spreadsheet gives solutions  $I_3 = 0.084211$  milliamps,  $I_2 = -0.23158$  milliamps and  $I_1 = -0.93158$  milliamps.

If we assume earth to be at 0v it follows that A will be at  $15(I_2 - I_3)$   
 $= 4.73684$ v.

It can be shown that the alternative route to A through the 6V battery will give  $+6 + 15I_3 = 4.73684$ v, this is a confirmation of our answer.

When the switch is thrown open,  $I_3$  ceases, observing loop2 we note

$(15 + 40)I_2 = -14$ ,  $I_2 = -0.24546$  milliamps. The potential at A is then

$$- (-0.24546 \times 15)(\text{k*milliamps}) = 3.6819\text{v.}$$

## 1.2 Linear programming.

### Example 4

Consider the problems faced by a factory production manager. His factory produces a range of 4 mobile phones, products P1, P2, P3 and P4 produced on 5 machines M1, M2, M3, M4 and M5.

For each unit produced of product P1, he must use M1 for 3 hours, M3 for 5 hours, M4 for 2 hours and M5 for 2 hours.

For each unit produced of product P2, he must use M1 for 2 hours, M2 for 2 hours, M3 for 3 hours and M5 for 2 hours.

For each unit produced of product P3, he must use M1 for 2 hours, M2 for 2 hours, M4 for 2 hours and M5 for 2 hours.

For each unit produced of product P4, he must use M1 for 2 hours, M2 for 1 hour, M3 for 1 hours, M4 for 1 hours and M5 for 1 hours.

The respective profits per unit produced are P1 £950, P2 £350, P3 £450 and P4 £200.

In any 1 week M1 runs for a maximum 192hours, M2 192 hours, M3 96 hours and M5 192 hours

- a) If he can sell all he produces, how much of each product would he produce and what would be his profit?
- b) If he has a confirmed order for 9 units of P1 that he must meet, how would he maximize profit on his remaining resources and what would be his product run?
- c) If he is allowed to covert 48 hours on any machine to 48 hours on another what exchange would he make to maximise his profits under constraints given in b).

To solve this problem, we return to spreadsheet LinEqSoln.xls, go to tab Lin Programming, you will find the page set up for the solution. To ease explanation I will refer to cells on the page by their cell references.

Row 7 lists the various production machines, column C lists the products by number. Row 8 lists the maximum working hours for each machine. Each product uses 2 rows, thus product 1 uses rows 12 & 13, product 2 uses rows 14 & 15 etc. Column D shows the entered value of nominal profit per item produced, the figure in column D second product cell enclosed with a bold border is the number of products we propose to produce. The first row of each product details the entered value of the number of hours per machine in the production of the product, thus cell G5 shows we require 5 hours on machine 3 to produce a unit of product 1. The second row of the product indicates the number of hours used on each machine to produce the proposed number of the product and column K shows the profit of producing that number of products. Cell K22 shows the total profit across all the products. The values of hours per machine per unit product, and profit for each product are user adjustable and so the table is easily adaptable for another problem.

The way the table has been set originally is to propose just 1 unit of each product. Looking at product 1 we see, as expected we have consumed 3 hours from M1, 5 hours from M3, 2 hours from M4, 2 hours from M5 and produced profit of £950. The total profit is £1950. Change cell D13 to 2, i.e. we want to produce 2 units of P1. We can immediately see in row 9 how the available hours for each machine are decreased, how the profit from P1 is increased to £1900 and how the total profit has been incremented to £2900.

This is obviously not a solution of our problem but it shows how the table works and how we can use it to produce solutions.



a) Product P1 has the largest profit per unit, if we want to produce this to a maximum it will be limited by machine 3. Try producing 19 units of P1. This uses 95 hours of M3 and leaves only enough time for 1 unit of P4, giving a total profit of £18250.

Product P1 seems to be consuming too many hours of M3 to give an optimum result, let us try 17 P1. The reduction in 2 units of P1 releases 10 hours on M3, this can be used by 5 units of P3. the resulting profit has increased to £18600.

P4 uses only 1 hour of M3 per unit, let us try producing 96 units. This produces a profit of £19200.

If we prioritise the production of P3 we can produce 48 units with a profit of £21600. This appears to be the maximum profit we can gain if we are unrestricted in production.

b) The required 9 units of P1 utilize 45 hours of M3, leaving just 51 hours for further use, this being the limiting constraint on further production. It is easily confirmed that under the constraints, 9 P1, 25 P3 and 1 P4 is the optimum giving a profit of £20000.

c) The results from b) show a residual 141 hours available on M2. Set the maximum hours available on M2 to 144 and increase the maximum available hours for M3 to 144. Under this regime, the 9 units of P1 use 45 hours on M3 leaving 99 hours. If we try to produce 45 units of P3, M4 becomes 20 hours in deficit, showing it is not feasible. We can only produce 39 units of P3. The remaining 21 hours of M3 to produce 7 units of P2. the solution is then 9 P1, 7 P2, 39 P3 and the profit is then increased to £28550.

## SUMMARY

We have seen in this extract, how I have demonstrated the use of mathematics for the quantitative solutions of commercial and industrial

problems. In this extract I have limited the complexity of the problems to give †tasters‡ to the possibilities and strength of the tools we are using. The two spreadsheets created and used in our solutions are unelaborate, but when understood are wonderfully powerful, demonstrating flexibility, accuracy and potential, it shows how these tools can be used to model processes and scenarios on a purely theoretical level predicting very potent outcomes.

In this article I have used a spreadsheet tool I have created in LinEqSoln.xls for solving linear equations to determine unit costs in Examples 1 and 2, in Example 3 I used the same tool to show how linear equations are used in solving dc current circuits. In Example 4 I have shown how a simple spreadsheet can model production scenarios and give immediate and easily understood results, facilitating the optimisation of resource usage.

I intend to go to greater depth with more sophisticated problems of more variety in a fuller unexpurgated version of this document that I hope to publish under Lulu.com in the second half of the coming year 2009. In this document I shall have examples of analyses of a more complicated dc circuit, analysis of ac circuits, pipe network analysis, optimisation of classical transport and logistic problems, three dimensional net drawing and transformations, vector analysis, trajectories of objects in a viscous fluid. I trust you will enjoy our walk through these applied mathematics applications.

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